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Quasiaxisymmetry near axisymmetry

The theoretical concept of quasisymmetry [1] explains how a stellarator can overcome the lack of a continuous symmetry in its magnetic field to achieve good confinement of collisionless particle orbits: a quasisymmetric equilibrium instead satisfies a hidden symmetry in special magnetic ("Boozer") coordinates. However, with some notable exceptions, most work on quasisymmetric stellarators has been based on numerical search algorithms, which find these special equilibria by sifting through parameter space. This approach, although the foundation of modern stellarator optimization, might be improved by advancing our basic understanding of the special set of stellarator equilibria that have good particle confinement – more generally referred to as "omnigenous" equilibria.

Garren and Boozer [2] performed an asymptotic expansion of the equations of magnetostatic equilibria near the magnetic axis, i.e., a large aspect ratio expansion. Based on this expansion, they argued that quasisymmetry cannot be satisfied globally and ultimately breaks down as the distance from the magnetic axis increases. As an alternative to satisfying quasisymmetry near the axis, they argued that there should also be sufficient freedom to satisfy the condition exactly on a single magnetic surface, at a finite distance from the axis. However, several open problems remained, such as the proof of the existence of such solutions, how numerous they are, and how to construct them. For practical purposes (e.g., to satisfy engineering constraints), and also to satisfy sheer curiosity, it would also be interesting to know of any geometric consequences of quasisymmetry – are there any basic limits to what sort of shapes are possible? The above problems also apply to the generalization of quasisymmetry (omnigeneity).

In our recent paper [3] we propose a new approach to generate quasiaxisymmetric (QAS) equilibria, a subclass of quasisymmetry [4], by deforming axisymmetric equilibria. We demonstrate the existence of a family of solutions near an arbitrarily specified axisymmetric zeroth-order solution that satisfy QAS on a single flux surface. Wes also prove

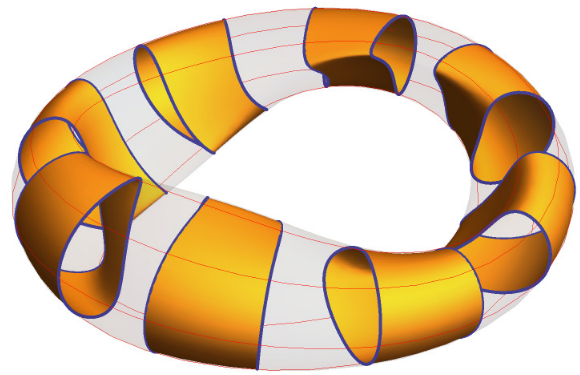

Fig. 1: Flux surface shape induced by quasiaxisymmetry-preserving perturbation (assuming circular zeroth-order shape).

that QAS cannot be satisfied globally (when axisymmetry is violated), supporting the conclusion of Ref. 2. The perturbed solutions can be found numerically by applying the condition of QAS as a (non-standard) boundary condition. Importantly, the solutions are valid globally (even though QAS is satisfied only on the outer surface), distinguishing them from solutions that only satisfy MHD locally [5, 6]. The more general class of omnigenous solutions is also shown to exist. Key parts of this work are summarized below.

In this issue . . .

Quasiaxisymmetry near axisymmetry
A new approach is proposed to generate optimized stellarator equilibria by deforming axisymmetric equilibria. A large set of solutions that satisfy quasiaxisymmetry (QAS) on a single magnetic flux surface is identified. Solutions are independently verified with a widely used MHD equilibrium solver, thereby demonstrating that QAS solutions can be approximately directly constructed, i.e., without using numerical search algorithms. 1

The first step is to reformulate the MHD equations in terms of a coordinate mapping $\boldsymbol{x}$, which is taken to be a function of the flux ψ, and the Boozer angles θ and φ. This "inverse" problem formulation allows the condition of QAS, namely that the field magnitude is independent of the Boozer toroidal angle φ, to be directly enforced. The coordinate mapping may be expressed in cylindrical coordinates as

$$\boldsymbol{x} = \hat{\mathbf{R}}R + \hat{\mathbf{z}}Z + \hat{\varphi}\Phi\,, \tag{1}$$

where we note that the component in the φ-direction is included for mathematical convenience. In terms of $\boldsymbol{x}$ the magnetostatic equilibrium equation in a vacuum can be written simply as

$$\frac{\partial \boldsymbol{x}}{\partial \varphi} + \frac{\iota}{2\pi}\frac{\partial \boldsymbol{x}}{\partial \theta} = G\frac{\partial \boldsymbol{x}}{\partial \psi}\times\frac{\partial \boldsymbol{x}}{\partial \theta}\,. \tag{2}$$

The solution is expanded as $\boldsymbol{x} = \boldsymbol{x}_0 + \varepsilon \boldsymbol{x}_1 + \varepsilon^2 \boldsymbol{x}_2 + \dots$, where $\boldsymbol{x}_0$ denotes the zeroth-order axisymmetric part, and Eq. (2) is solved, order by order. Due to symmetry, the toroidal angle is ignorable in the first-order equations, and the mode number N is introduced as a free parameter. With some manipulation of Eq. (2), the problem can then be transformed into a remarkably simple form, a single second-order elliptic partial differential equation for $P_1(R_0,Z_0)$, the first-order part of the toroidal component Φ:

$$(N^2 - 1)P_1 = R_0^2 \Delta_0^* P_1\,. \tag{3}$$

We note that the equation is differential in the zeroth-order cylindrical coordinates R_0 and Z_0, and define the Grad-Shafranov operator in these coordinates,

$$\Delta_0^* = R_0\frac{\partial}{\partial R_0}\left(\frac{1}{R_0}\frac{\partial}{\partial R_0}\right) + \frac{\partial^2}{\partial Z_0^2}\,. \tag{4}$$

The solution of Eq. 3 determines the entire magnetic field solution at first order.The QAS condition ($\partial B/\partial\varphi=0$) at this order, can be written as simply

$$(N^2 - 1)P_1 + R_0\frac{\partial P_1}{\partial R_0} = 0\,. \tag{5}$$

Note that the equations are to be solved in the R_0-Z_0 plane within a domain defined by the zeroth-order axisymmetric solution. This means that, although the ultimate shape of the outer surface (i.e., including the deformation) is *a priori* unknown, the problem is still of the fixed boundary type. A series of results follow rather directly from Eqs. (3) and (5). First, it is found that the equations cannot be both satisfied across the entire volume (R_0-Z_0) plane, except for trivial solutions that preserve axisymmetry. This fact is shown by solving Eq. (5), a first order ODE, and substituting the result into Eq. (3). Second, it is noted that Eq. (5) when applied on the boundary of the domain, is an oblique-derivative boundary condition; this condition was first studied by Poincaré [7]. The existence of solutions in the two-dimensional case was proved in the 1950s [8], and the results are summarized in [9]. For the particular case here, two linearly independent solutions are guaranteed to exist.

We can thus conclude that the space of weakly non-axisymmetric QAS equilibria can be parameterized by (1) the axisymmetric surface shape (2D function), (2) the toroidal mode number N (to be interpreted as the field-period number), and (3) two complex numbers corresponding to the solution space of the oblique-derivative problem. To independently confirm the theoretical results, these numerical solutions are used as input for the VMEC equilibrium solver [10], coupled to the BOOZ_XFORM code [11] which recovers the Boozer-coordinate representation. For the purposes of this test, we assume $N = 2$ and take the zeroth-order flux surface to have a circular shape, and an aspect ratio of 4. The strength of the perturbation is controlled by the free parameter ε, and as an example, the resulting flux surface shape for the case considered is depicted in Fig. 1. The satisfaction of QAS is thereby confirmed at the appropriate order; see Fig. 2.

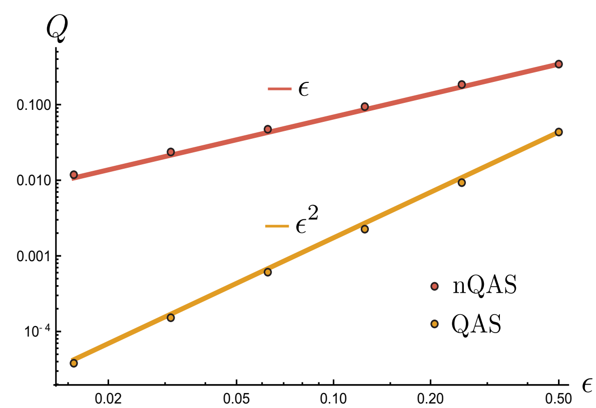

Fig. 2. Test of QAS at first order in size of non-axisymmetric perturbation; note the measure

$$Q = \frac{\left(\sum_{m,n\neq\mathbf{0}} |\hat{B}_{mn}|^2\right)^{1/2}}{\left(\sum_{m,n} |\hat{B}_{mn}|^2\right)^{\mathbf{1}/2}}\,,$$

where $\hat{B}$ is the Fourier amplitude of ⊠(θ,φ) at the outer flux surface. Solutions of the problem (labeled QAS) are compared with a "control" deformation (nQAS), yielding ε^2 and ε scaling, respectively, as expected; see Ref. [3] for further details.

Omnigeneity, the general condition that collisionless particle orbits are (radially) confined, is also considered as a boundary condition for the deformation. Omnigeneity can be restated geometrically as the condition that the distance along a field line from two points of equal magnetic field strength does not vary between neighboring field lines [12]. The existence of such solutions, satisfying omnigeneity on one surface, is also guaranteed for the associated oblique derivative problem [9]. Finally, we note that the QAS perturbations represent a freedom in the omnigenous problem, as they can be superposed onto the omnigenous solution without affecting the satisfaction of the condition of omnigeneity. These encouraging results demonstrate that it is possible to numerically construct omnigenous solutions, using an approach similar to that used for QAS ones, though this is left for future work.

References

[1] J. Nührenberg and R. Zille. Quasi-helically symmetric toroidal stellarators. Phys. Lett. A **129** (2):113 – 117, 1988. ISSN 0375-9601.

[2] D. A. Garren and A. H. Boozer. Existence of quasihelically symmetric stellarators. Phys. Fluids B **3** (10):2822–2834, 1991.

[3] G. G. Plunk and P. Helander. Quasiaxisymmetric equilibria: weakly nonaxisymmetric case in a vacuum. ArXiv e-prints; submitted to J. Plasma Phys., January 2018.

[4] J. Nührenberg, E. Sindoni, W. Lotz, F. Troyon, S. Gori, and J. Vaclavik. Quasi-axisymmetric tokamaks. In Proc. Joint Varenna-Lausanne International Workshop on Theory of Fusion Plasmas, pp 3–12, 1994.

[5] C. C. Hegna. Local three-dimensional magnetostatic equilibria. Phys. Plasmas **7** (10):3921–3928, 2000.

[6] Allen H. Boozer. Local equilibrium of nonrotating plasmas. Phys. Plasmas, **9** (9):3762–3766, 2002.

[7] Jules Henri Poincaré, Leçons de Mécanique Céleste, Tome III, Théorie de Marees. Cours de la faculté des sciences de Paris. Gauthier-Villars, Paris, 1910.

[8] I. N. Vekua. A boundary problem with oblique derivative for an equation of elliptic type. Dokl. Akad. Nauk SSSR (NS), 92, 1953.

[9] Carlo Miranda. Partial Differential Equations of Elliptic Type. Springer-Verlag New York Inc., 1970. ISBN 978-3-642- 87775-9.

[10] S. P. Hirshman and J. C. Whitson. Steepest-descent moment method for three-dimensional magnetohydrodynamic equilibria. Phys. Fluids **26** (12):3553–3568, 1983.

[11] R. Sanchez, S. P. Hirshman, A. S. Ware, L. A. Berry, and D. A. Spong. Ballooning stability optimization of low-aspect-ratio stellarators. Plasma Phys. Control. Fusion **42** (6):641, 2000.

[12] John R. Cary and Svetlana G. Shasharina. Omnigenity and quasihelicity in helical plasma confinement systems. Phys. Plasmas **4** (9):3323–3333, 1997, http://dx.doi.org/10.1063/1.872473

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